Spontaneous symmetry breaking in the *gl*(*N*)-NLS hierarchy on the half line

2001 J. Phys. A: Math. Gen. 34 8345

(http://iopscience.iop.org/0305-4470/34/40/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8345–8364 PII: S0305-4470(01)24954-3

Spontaneous symmetry breaking in the $gl(N)$-NLS hierarchy on the half line

M Mintchev[1], **E Ragoucy**[2] **and P Sorba**[2]

[1] INFN, Dipartimento di Fisica dell'Univ. di Pisa, Via Buonarroti 2, 56127 Pisa, Italy
[2] LAPTH, Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France

E-mail: mintchev@df.unipi.it, ragoucy@lapp.in2p3.fr and sorba@lapp.in2p3.fr

Received 11 May 2001, in final form 20 August 2001
Published 28 September 2001
Online at stacks.iop.org/JPhysA/34/8345

Abstract
We describe an algebraic framework for studying the symmetry properties of integrable quantum systems on the half line. The approach is based on the introduction of boundary operators. It turns out that these operators both encode the boundary conditions and generate integrals of motion. We use this direct relationship between boundary conditions and symmetry content to establish the spontaneous breakdown of some internal symmetries, due to the boundary.

PACS numbers: 02.30.Ik, 02.20.-a, 03.65.-w, 03.70.+k, 11.10.-z, 11.30.-j

1. Introduction

This paper describes a framework for studying the symmetry properties of integrable quantum systems on the half line. As can be expected on general grounds, the presence of a boundary in this case has a strong impact on the dynamics and the symmetry content of the systems. It gives rise to a variety of boundary-related phenomena with direct applications to impurity problems in condensed matter, dissipative quantum mechanics, open string theory and brane physics. The recent efforts to gain a deeper insight into these phenomena stimulated a series of investigations [1–9] in the subject. Among others, one should mention the attempts to develop an algebraic approach. The new strategy there is to introduce [1, 2] the so-called *boundary operators*, which translate into algebraic terms the associated quantum boundary value problem. It is far from being a surprise that, if possible, an algebraic treatment of the boundary value problem turns out to be simpler than the standard analytic one. In the context of integrable systems, the algebraic framework provides a further advantage—the search for general integrability preserving boundary conditions and their implementation becomes straightforward.

The concept of boundary algebra, which we will be dealing with below, has been introduced in [10] and is inspired by Cherednik's scattering theory [11] for integrable models on the half line. An essential feature of this algebra, denoted in what follows by $\mathcal{B}_R$, is that it enables one to reconstruct [10] not only the scattering matrix, but also captures the off-shell dynamics (correlation functions) of the system [12, 13]. Therefore, one can apply $\mathcal{B}_R$ in the study of symmetries as well. This observation is the basic starting point of our investigation. The most interesting property emerging from it is that, besides encoding the boundary conditions, the boundary operators also generate integrals of motion. The main goal of the present paper is to investigate this remarkable feature and to explore the consequences from it.

The paper is organized as follows. In the next section we summarize those basic features of the boundary algebra $\mathcal{B}_R$ and its Fock representations, which are needed for our analysis. The general structure involved in the discussion is illustrated by means of the $gl(N)$-invariant nonlinear Schrödinger model on the half line. The quantization of the model in terms of $\mathcal{B}_R$ is briefly described in section 3. Section 4 is devoted to a purely algebraic analysis of the symmetry content of the NLS model. In section 5 we derive a large class of boundary conditions which preserve integrability. The phenomenon of spontaneous symmetry breaking is established in section 6. Section 7 contains a detailed study of the case $N = 2$. Finally, section 8 collects our conclusions.

2. Boundary algebras

We recall first the basic structure of a boundary algebra, referring for details to [10]. $\mathcal{B}_R$ is an associative algebra with identity element $\mathbf{1}$ and two types of generators:

$$\{a_i(k),\ a^{*i}(k)\ :\ i = 1, \ldots, N,\ k \in \mathbb{R}\} \tag{2.1}$$

and

$$\{b_i^j(k)\ :\ i, j = 1, \ldots, N,\ k \in \mathbb{R}\} \tag{2.2}$$

called bulk and boundary generators, respectively. N is the number of internal degrees of freedom, whereas k stands for the momentum in the non-relativistic case or the rapidity in the relativistic one. Equations (2.1) and (2.2) are subject to the following constraints:

(i) bulk exchange relations:

$$a_{i_1}(k_1)\ a_{i_2}(k_2) - R_{i_2 i_1}^{j_1 j_2}(k_2, k_1)\ a_{j_2}(k_2)\, a_{j_1}(k_1) = 0 \tag{2.3}$$

$$a^{*i_1}(k_1)\, a^{*i_2}(k_2) - a^{*j_2}(k_2)\, a^{*j_1}(k_1)\, R_{j_2 j_1}^{i_1 i_2}(k_2, k_1) = 0 \tag{2.4}$$

$$a_{i_1}(k_1)\, a^{*i_2}(k_2) - a^{*j_2}(k_2)\, R_{i_1 j_2}^{i_2 j_1}(k_1, k_2)\, a_{j_1}(k_1) = 2\pi\ \delta(k_1 - k_2)\, \delta_{i_1}^{i_2}\, \mathbf{1} + 2\pi\ \delta(k_1 + k_2)\, b_{i_1}^{i_2}(k_1) \tag{2.5}$$

(ii) boundary exchange relations:

$$R_{i_1 i_2}^{l_2 l_1}(k_1, k_2)\, b_{l_1}^{m_1}(k_1)\, R_{l_2 m_1}^{j_1 m_2}(k_2, -k_1)\, b_{m_2}^{j_2}(k_2) = b_{i_2}^{l_2}(k_2)\, R_{i_1 l_2}^{m_2 m_1}(k_1, -k_2)\, b_{m_1}^{l_1}(k_1)\, R_{m_2 l_1}^{j_1 j_2}(-k_2, -k_1) \tag{2.6}$$

(iii) mixed exchange relations:

$$a_{i_1}(k_1)\, b_{i_2}^{j_2}(k_2) = R_{i_2 i_1}^{l_1 l_2}(k_2, k_1)\, b_{l_2}^{m_2}(k_2)\, R_{l_1 m_2}^{j_2 m_1}(k_1, -k_2)\, a_{m_1}(k_1) \tag{2.7}$$

$$b_{i_1}^{j_1}(k_1)\, a^{*i_2}(k_2) = a^{*m_2}(k_2)\, R_{i_1 m_2}^{l_2 m_1}(k_1, k_2)\, b_{m_1}^{l_1}(k_1)\, R_{l_2 l_1}^{j_1 i_2}(k_2, -k_1). \tag{2.8}$$

Here and in what follows the summation over repeated upper and lower indices is always understood. Finally, using already standard notations, the exchange factor R obeys

$$R_{12}(k_1, k_2)\, R_{12}(k_2, k_1) = 1 \tag{2.9}$$

and the spectral quantum Yang–Baxter equation (in its braid form)

$$R_{12}(k_1, k_2)R_{23}(k_1, k_3)R_{12}(k_2, k_3) = R_{23}(k_2, k_3)R_{12}(k_1, k_3)R_{23}(k_1, k_2). \tag{2.10}$$

These compatibility conditions on R, together with equations (2.3)–(2.8), define the algebra $\mathcal{B}_R$. Let us observe also that the generators $\{b_i^j(k)\}$ close a subalgebra $\mathcal{S}_R \subset \mathcal{B}_R$, which has been introduced by Sklyanin [14] and which will play a distinguished role in our discussion below.

It is worth mentioning that, if one formally takes $b_i^j(k) \to 0$, the relations (2.6)–(2.8) trivialize, while (2.3)–(2.5) reproduce the defining relations of a Zamolodchikov–Faddeev (ZF) algebra [15, 16], which is known to describe factorized scattering of $(1+1)$-dimensional integrable systems on the whole line. The situation changes on the half line, where in order to reproduce Cherednik's scattering amplitudes one needs [10] a *reflection algebra*, i.e. a $\mathcal{B}_R$ with the additional constraint

$$b_i^l(k)\, b_l^j(-k) = \delta_i^j \tag{2.11}$$

which obviously prevents the vanishing of all boundary operators. The condition (2.11) has important consequences; it implies that the mapping

$$\varrho \;:\; a_i(k) \mapsto b_i^j(k) a_j(-k) \tag{2.12}$$

$$\varrho \;:\; a^{*i}(k) \mapsto a^{*j}(-k) b_j^i(-k) \tag{2.13}$$

$$\varrho \;:\; b_i^j(k) \mapsto b_i^j(k) \tag{2.14}$$

leaves invariant the constraints (2.3)–(2.8) and extends therefore to an automorphism on $\mathcal{B}_R$. ϱ, which is called in what follows *reflection automorphism*, has a direct physical interpretation in scattering theory: it provides a mathematical description of the intuitive physical picture that, bouncing back from a wall, particles change the sign of their momenta (rapidities). In fact, the two elements $a^{*i}(-k)$ and $a^{*j}(k)b_j^i(k)$ are ϱ-equivalent:

$$a^{*i}(-k) \sim a^{*j}(k) b_j^i(k). \tag{2.15}$$

Combined with (2.6), equation (2.11) turns out to also be essential in the description of symmetries.

For constructing representations of $\mathcal{B}_R$, it is essential to recognize an involution on it. Let us consider for this purpose the mapping I defined by

$$I \;:\; a^{*i}(k) \mapsto a_i(k) \qquad a_i(k) \mapsto a^{*i}(k) \qquad b_i^j(k) \mapsto b_j^i(-k). \tag{2.16}$$

When extended as an antilinear antihomomorphism on $\mathcal{B}_R$, I defines an involution, provided that R satisfies

$$R^{\dagger j_1 j_2}_{i_1 i_2}(k_1, k_2) = R^{j_1 j_2}_{i_1 i_2}(k_2, k_1). \tag{2.17}$$

Here and in what follows the dagger stands for matrix Hermitian conjugation, i.e.

$$R^{\dagger j_1 j_2}_{i_1 i_2}(k_1, k_2) \equiv \overline{R}^{i_1 i_2}_{j_1 j_2}(k_1, k_2) \tag{2.18}$$

the bar indicating complex conjugation. The condition (2.17) is known in the literature on factorized scattering as *Hermitian analyticity*. We observe in passing that a larger class of involutions with the appropriate generalization of (2.17) has been introduced in [10].

Since $\mathcal{B}_R$ is an infinite algebra, for the moment our considerations are a bit formal. In order to give them a precise mathematical meaning, one can construct [10] a class of Fock representations of the reflection algebra with involution $\{\mathcal{B}_R, I\}$, characterized by the following requirements:

(1) The representation space $\mathcal{H}$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$;
(2) The generators (2.1) and (2.2) are operator valued distributions with common and invariant dense domain $\mathcal{D} \subset \mathcal{H}$, where equations (2.3)–(2.8) hold. The involution I defined by equation (2.16) is realized as a conjugation with respect to $\langle \cdot, \cdot \rangle$;
(3) There exists a vector (vacuum state) $\Omega \in \mathcal{D}$ which is annihilated by $a_i(k)$. Ω is cyclic with respect to $\{a^{*i}(k)\}$ and $\langle \Omega\, , \, \Omega \rangle = 1$.

These requirements imply that the c-number distributions:

$$B_i^j(k) \equiv \langle \Omega\, , \, b_i^j(k)\Omega \rangle \tag{2.19}$$

satisfy

$$B^{\dagger j}_{\ i}(k) = B_i^j(-k) \tag{2.20}$$

which, being an analogue of (2.17), is called *boundary* Hermitian analyticity. One can show moreover that the vacuum vector Ω is unique (up to a phase factor) and satisfies

$$b_i^j(k)\, \Omega = B_i^j(k)\, \Omega. \tag{2.21}$$

Therefore, taking the vacuum expectation value of equations (2.6) and (2.11), one finds

$$\begin{aligned} &R_{i_1 i_2}^{l_2 l_1}(k_1, k_2)\, B_{l_1}^{m_1}(k_1)\, R_{l_2 m_1}^{j_1 m_2}(k_2, -k_1)\, B_{m_2}^{j_2}(k_2) \\ &\qquad = B_{i_2}^{l_2}(k_2)\, R_{i_1 l_2}^{m_2 m_1}(k_1, -k_2)\, B_{m_1}^{l_1}(k_1)\, R_{m_2 l_1}^{j_1 j_2}(-k_2, -k_1) \end{aligned} \tag{2.22}$$

$$B_i^l(k)\, B_l^j(-k) = \delta_i^j. \tag{2.23}$$

We thus recover at the level of Fock representation the *boundary* Yang–Baxter equation (2.22), originally derived in [11]. Because of (2.23), we will refer to B as the *reflection matrix*.

Given a reflection algebra $\{\mathcal{B}_R, I\}$, its Fock representations are classified by all possible reflection matrices B, which actually parametrize the boundary conditions. More precisely, any B-matrix satisfying equations (2.20), (2.22) and (2.23) defines a Fock representation $\mathcal{F}_B$, whose explicit construction is given in [10]. Each $\mathcal{F}_B$ uniquely defines in turn a unitary scattering operator S, corresponding to the integrable model described by $\{\mathcal{B}_R, I\}$. The emerging picture is therefore the following. The mere fact that we are dealing with an integrable system with boundary shows up at the algebraic level, turning the ZF algebra into a reflection algebra. The details of the boundary condition enter at the representation level through the reflection matrix B. We stress at this point another sharp difference between reflection and ZF algebras, the latter admitting a unique (up to unitary equivalence) Fock representation.

Let us turn finally to the question of symmetries, considering for instance the simplest non-relativistic Hamiltonian that comes to mind, namely

$$H = \int_{-\infty}^{\infty} \mathrm{d}p\; p^2 a^{*i}(p) a_i(p). \tag{2.24}$$

Using equations (2.7) and (2.8), one easily verifies that

$$\left[H\, , \, b_i^j(k)\right] = 0. \tag{2.25}$$

Therefore, besides capturing the presence of boundaries, $b_i^j(k)$ generate integrals of motion, whose algebra is encoded in the boundary exchange relations (2.6).

It will be our main objective in what follows to examine thoroughly this observation on both purely algebraic (section 4) and Fock representation (section 5) levels. In order to gain some intuition from a concrete example, we find it useful to illustrate first the above abstract setup by means of the $gl(N)$-invariant nonlinear Schrödinger ($gl(N)$-NLS) model on the half line.

Note also that the property (2.25) can be extended to a wide class of operators H_n:

$$[H_n,\, b_i^j(k)] = 0 \qquad \text{with} \quad H_n = \int_{-\infty}^{\infty} \mathrm{d}p\; p^n a^{*i}(p) a_i(p) \tag{2.26}$$

leading to the notion of hierarchy. However, we want to stress that one has to check the vanishing of the commutators between different H_n to get a true hierarchy. A direct calculation shows

$$[H_n, H_m] = \left((-1)^n - (-1)^m\right) \int_{-\infty}^{\infty} \mathrm{d}p\; p^{m+n} a^{*i}(-p) b_i^j(-p) a_j(p). \tag{2.27}$$

Thus, it is only the H_n of same parity that commute one with each other, and the hierarchy should be associated with the 'even' Hamiltonians H_{2n}. The hierarchy associated with the 'odd' Hamiltonians H_{2n+1} is trivial in the Fock representation, because there

$$H_{2n+1} = 0. \tag{2.28}$$

In what follows, we will concentrate on the Hamiltonian H_2, but one has to keep in mind that the properties will apply to the whole hierarchy $\{H_{2n},\ n \in \mathbb{N}\}$.

3. The NLS model on the half line

The dynamics of the $gl(N)$-NLS model can be described by a N-component field $\Psi_i(t, x)$, satisfying

$$(\mathrm{i}\partial_t + \partial_x^2)\Psi_i(t,x) = 2g\, \Psi^{\dagger j}(t,x)\Psi_j(t,x)\Psi_i(t,x) \qquad g > 0 \tag{3.1}$$

on the half line $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. One must fix in addition the boundary conditions. We start by requiring

$$\lim_{x\downarrow 0}(\partial_x - \eta)\Psi_i(t,x) = 0 \qquad \eta \geqslant 0 \tag{3.2}$$

$$\lim_{x\to\infty} \Psi_i(t,x) = 0. \tag{3.3}$$

Equation (3.2) is the so-called mixed boundary condition; for $\eta = 0$ and in the limit $\eta \to \infty$ it reproduces the familiar Neumann and Dirichlet conditions, respectively. We will focus in this section on the boundary value problem (3.1)–(3.3), postponing the consideration of more general integrability preserving boundary conditions to section 4.

In spite of the fact that the NLS model is among the most studied integrable systems, to our knowledge there only exist a few papers addressing the problem on $\mathbb{R}_+$. References [14] and [17] deal essentially with the proof of integrability, whereas [12] and [13] are concerned with the construction of the quantum field Ψ and the relative off-shell correlation functions. We will follow [12, 13] below, recalling first the basic structures which enter the second quantization of the boundary value problem (3.1)–(3.3):

(1) A Hilbert space $\{\mathcal{H}_{g,\eta},\ \langle\cdot,\cdot\rangle\}$ describing the states of the system;
(2) An operator valued distribution $\Psi(t,x)$ which acts on an appropriate dense domain $\mathcal{D} \subset \mathcal{H}_{g,\eta}$ and satisfies:
   (a) the canonical commutation relations:

$$[\Psi_i(t,x), \Psi_j(t,y)] = [\Psi^{*i}(t,x), \Psi^{*j}(t,y)] = 0 \tag{3.4}$$

$$[\Psi_i(t,x), \Psi^{*j}(t,y)] = \delta_i^j\, \delta(x-y) \tag{3.5}$$

   where Ψ^* is the $\langle\cdot,\cdot\rangle$-Hermitian conjugate of Ψ;
   (b) the equation of motion:

$$(\mathrm{i}\partial_t + \partial_x^2)\langle\varphi, \Psi_i(t,x)\psi\rangle = 2g\,\langle\varphi, :\Psi_j\Psi^{*j}\Psi_i:(t,x)\psi\rangle \qquad \forall\, \varphi, \psi \in \mathcal{D} \tag{3.6}$$

   where $: \ldots :$ denotes a suitably defined normal product;

(c) the boundary conditions:

$$\lim_{x\downarrow 0}\,(\partial_x - \eta)\,\langle \varphi\,,\ \Psi_i(t,x)\psi\rangle = 0 \qquad \forall\ \varphi,\psi\in\mathcal{D} \tag{3.7}$$

$$\lim_{x\to\infty}\langle \varphi\,,\ \Psi_i(t,x)\psi\rangle = 0 \qquad \forall\ \varphi,\psi\in\mathcal{D} \tag{3.8}$$

(3) A fundamental state $\Omega\in\mathcal{D}$, which is cyclic with respect to Ψ^*.

It has been shown in [13] that all these structures can be explicitly realized in terms of the reflection algebra $\mathcal{B}_R$ defined by the exchange matrix:

$$R_{i_2 i_1}^{j_1 j_2}(k_1,k_2) = \frac{1}{k_1-k_2+\mathrm{i}g}\left[\mathrm{i}g\,\delta_{i_1}^{j_1}\,\delta_{i_2}^{j_2} + (k_1-k_2)\,\delta_{i_1}^{j_2}\,\delta_{i_2}^{j_1}\right] \tag{3.9}$$

and the associated Fock representation $\mathcal{F}_B$ with reflection matrix:

$$B_i^j(k) = \frac{k-\mathrm{i}\eta}{k+\mathrm{i}\eta}\,\delta_i^j. \tag{3.10}$$

One has to identify for this purpose $\mathcal{H}_{g,\eta}$ with the Fock space, Ω with the Fock vacuum and $\langle\cdot,\cdot\rangle$ with the scalar product of the representation $\mathcal{F}_B$. Moreover, the quantum field Ψ admits the expansion

$$\Psi_i(t,x) = \sum_{n=0}^{\infty}(-g)^n\,\Psi_i^{(n)}(t,x) \tag{3.11}$$

where

$$\Psi_i^{(n)}(t,x) = \int_{-\infty}^{\infty}\prod_{\substack{i=1\\ j=0}}^{n}\frac{\mathrm{d}p_i}{2\pi}\,\frac{\mathrm{d}q_j}{2\pi}\,\frac{\mathrm{e}^{\mathrm{i}\sum_{j=0}^{n}(xq_j-tq_j^2)-\mathrm{i}\sum_{i=1}^{n}(xp_i-tp_i^2)}}{\prod_{i=1}^{n}\left[(p_i-q_{i-1}-\mathrm{i}\epsilon)\,(p_i-q_i-\mathrm{i}\epsilon)\right]}$$
$$\times a^{*j_1}(p_1)\ldots a^{*j_n}(p_n)\,a_{j_n}(q_n)\ldots a_{j_1}(q_1)a_i(q_0) \tag{3.12}$$

a_i and a^{*j} representing the $\mathcal{B}_R$ generators in $\mathcal{F}_B$. By construction, the domain $\mathcal{D}$ involves only vectors with finite, although arbitrary large, particle number. Combining this property with the normal ordered form of $\Psi^{(n)}$, one concludes that the series (3.11) converges in mean value on $\mathcal{D}$.

Equation (3.12) is the quantum inverse scattering transform for the $gl(N)$-NLS model on $\mathbb{R}_+$. The field Ψ admits (strong) asymptotic limits, giving raise to the following asymptotic states:

$$|k_1,i_1;\ldots;k_n,i_n\rangle^{\mathrm{in}} = a^{*i_1}(k_1)\ldots a^{*i_n}(k_n)\Omega \qquad k_1<\cdots<k_n<0 \tag{3.13}$$

$$|p_1,j_1;\ldots;p_m,j_m\rangle^{\mathrm{out}} = a^{*j_1}(p_1)\ldots a^{*j_m}(p_m)\Omega \qquad p_1>\cdots>p_n>0. \tag{3.14}$$

The vectors (3.13) and (3.14) generate the asymptotic in- and out-spaces, respectively. Both of these spaces are dense in $\mathcal{H}_{g,\eta}$ and the model is asymptotically complete in the range $g>0$ and $\eta\geqslant 0$. The connection with Cherednik's scattering theory [11] is obtained on the level of scattering amplitudes:

$$^{\mathrm{out}}\langle p_1,j_1,\ldots,p_m,j_m|k_1,i_1,\ldots,k_n,i_n\rangle^{\mathrm{in}}. \tag{3.15}$$

The explicit form of (3.15) is easily derived and involves a product of R and B factors: any R factor describes a two-body scattering in the bulk $\mathbb{R}_+$, while the B factors take into account the reflection from the boundary.

It is worth stressing that the time evolution of the field $\Psi(t,x)$ is generated precisely by the Hamiltonian (2.24). In fact, by means of equation (3.12) one gets

$$\Psi_i(t,x) = \mathrm{e}^{\mathrm{i}tH}\,\Psi_i(0,x)\,\mathrm{e}^{-\mathrm{i}tH}. \tag{3.16}$$

Therefore, according to our discussion at the end of the previous section, the boundary operators $b_i^j(k)$ generate integrals of motion for the $gl(N)$-NLS model on $\mathbb{R}_+$. The relative algebra, which follows from equation (2.6) by inserting the specific exchange factor (3.9), will be investigated in the next section.

In conclusion, we point out that the quantum field (3.11), (3.12) solves (3.6) for any reflection matrix B satisfying equation (2.22) with R given by (3.9). One can also demonstrate [13] that, for any $\varphi,\ \psi\ \in\ \mathcal{D}$, there exist N square integrable functions $\chi_i(k)$, such that

$$\langle\varphi\,,\ \Psi_i(t,x)\psi\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{-\mathrm{i}tk^2}\left[\delta_i^j\,\mathrm{e}^{\mathrm{i}xk} + B_i^j(k)\mathrm{e}^{-\mathrm{i}xk}\right]\chi_j(k). \tag{3.17}$$

This identity implies not only (3.7) and (3.8), but allows us to analyse (see section 5) the boundary condition at $x = 0$ for general B factors.

4. The NLS symmetry algebra

Now we bring our attention to the $\mathcal{S}_R$ subalgebra defined from the boundary generators (2.2) and exchange relation (2.6), when the R-matrix is chosen as in (3.9). In the standard notation, the R-matrix takes the form:

$$R_{12}(k) = \frac{1}{k+\mathrm{i}\,g}\,(k\,\mathbb{I}\otimes\mathbb{I} + \mathrm{i}g\,P_{12}) \tag{4.1}$$

where $\mathbb{I}$ is the $N\times N$ identity matrix, $P_{12}\equiv P = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji}$ is the flip operator, and E_{ij} is the $N\times N$ matrix with 1 at position (i,j).

4.1. Calculation of the $\mathcal{S}_R$ algebra

Let us rewrite (2.6) as

$$R(k_1-k_2)\,b_1(k_1)\,R(k_1+k_2)\,b_2(k_2) = b_2(k_2)\,R(k_1+k_2)\,b_1(k_1)\,R(k_1-k_2) \tag{4.2}$$

or as commutation relation

$$\begin{aligned}[b_1(k_1), b_2(k_2)] &= \frac{\mathrm{i}g}{k_1-k_2}(b_2(k_2)b_1(k_1)P - Pb_1(k_1)b_2(k_2))\\ &\quad+\frac{\mathrm{i}g}{k_1+k_2}(b_2(k_2)Pb_1(k_1) - b_1(k_1)Pb_2(k_2)) - \frac{g^2}{k_1^2-k_2^2}\,[b_2(k_2), b_2(k_1)]\end{aligned} \tag{4.3}$$

where

$$b_1(k) = \sum_{i,j=1}^{N} b_{ij}(k)\,E_{ij}\otimes\mathbb{I} \qquad\text{and}\qquad b_2(k) = \sum_{i,j=1}^{N} b_{ij}(k)\,\mathbb{I}\otimes E_{ij}. \tag{4.4}$$

Note that it enters in the algebras of type $ABCD$, introduced in [18], and as such is well defined, since it obeys the corresponding consistency conditions (see [18] for details).

We develop $b(k)$ as a formal power series in k^{-1}:

$$b(k) = \sum_{n\geqslant 0} k^{-n} b^{(n)} = \sum_{n\geqslant 0}\sum_{i,j=1}^{N} k^{-n} b_{ij}^{(n)} E_{ij}. \tag{4.5}$$

Plugging this expression into (4.2) leads to the following commutation relations:

$$\begin{aligned}\sum_{m,n=0}^{\infty}[b_1^{(m)}, b_2^{(n)}]\,k_1^{-m}k_2^{-n} &= \sum_{p,q,r=0}^{\infty}\Big\{\mathrm{i}g\Big(b_2^{(q)}b_1^{(p)}P - Pb_1^{(p)}b_2^{(q)}\\ &\quad+(-1)^r\Big(b_2^{(q)}Pb_1^{(p)} - b_1^{(p)}Pb_2^{(q)}\Big)\Big)k_1^{-p-r-1}k_2^{r-q}\\ &\quad-g^2\left[b_2^{(q)}, b_2^{(p)}\right]k_1^{-p-2r-2}k_2^{2r-q}\Big\}.\end{aligned} \tag{4.6}$$

In order for the $\mathcal{S}_R$ algebra to be well-defined, no positive power of k can be admitted to the r.h.s. of the above expression. A direct computation shows that spurious terms are avoided by the necessary and sufficient conditions:

$$b(k)b(-k) = f(k)\,\mathbb{I} \qquad \text{with} \quad f(k) = 1 + \sum_{m=1}^{\infty} f_{2m} k^{-2m} \text{ even function} \tag{4.7}$$

$$[b_1^{(0)}, b_2(k)] = 0 \qquad \text{i.e. all the } b_{ij}^{(0)} \text{ generators are central.} \tag{4.8}$$

We remark that the condition (4.7) is a natural generalization of the physical condition (2.11) obtained in section 2. In the following, we will take $f(k)$ as a pure $\mathbb{C}$-function. The normalization of $f(k)$ is fixed by $\lim_{k\to\infty} f(k) = 1$.

Before commenting on these constraints, let us note that the commutation relations can then be rewritten as

$$\begin{aligned}[b_1^{(m)}, b_2^{(n)}] = \mathrm{i}g \sum_{r=0}^{m-1} \Big\{ & \left(b_2^{(n+r)} b_1^{(m-1-r)} - b_2^{(m-1-r)} b_1^{(n+r)} \right) P \\ & + (-1)^r \left(b_2^{(n+r)} P b_1^{(m-1-r)} - b_1^{(m-1-r)} P b_2^{(n+r)} \right) \Big\} \\ & - g^2 \sum_{r=0}^{\mu} \left[b_2^{(n+2r)}, b_2^{(m-2-2r)} \right] \qquad \text{where} \quad \mu = \left[\frac{m-2}{2} \right]\end{aligned} \tag{4.9}$$

that is

$$\begin{aligned}[b_{ij}^{(m)}, b_{kl}^{(n)}] = \mathrm{i}g \sum_{r=0}^{m-1} \Big\{ & \left(b_{kj}^{(n+r)} b_{il}^{(m-1-r)} - b_{kj}^{(m-1-r)} b_{il}^{(n+r)} \right) \\ & + (-1)^r \left(\delta_{il}\, b_{ka}^{(n+r)} b_{aj}^{(m-1-r)} - \delta_{kj}\, b_{ia}^{(m-1-r)} b_{al}^{(n+r)} \right) \Big\} \\ & - g^2 \sum_{r=0}^{\mu} \delta_{ij} \left(b_{ka}^{(n+2r)} b_{al}^{(m-2-2r)} - b_{ka}^{(m-2-2r)} b_{al}^{(n+2r)} \right)\end{aligned} \tag{4.10}$$

and also

$$\begin{aligned}[b_{ij}(k_1), b_{kl}(k_2)] = & \frac{\mathrm{i}g}{k_1 - k_2} \left(b_{kj}(k_2) b_{il}(k_1) - b_{kj}(k_1) b_{il}(k_2) \right) \\ & + \frac{\mathrm{i}g}{k_1 + k_2} \left(\delta_{il}\, b_{ka}(k_2) b_{aj}(k_1) - \delta_{kj}\, b_{ia}(k_1) b_{al}(k_2) \right) \\ & - \frac{g^2}{k_1^2 - k_2^2}\, \delta_{ij}\, \left(b_{ka}(k_2) b_{al}(k_1) - b_{ka}(k_1) b_{al}(k_2) \right).\end{aligned} \tag{4.11}$$

In particular, for $m = 1$, we get

$$[b_{ij}^{(1)}, b_{kl}^{(n)}] = \mathrm{i}g \left(b_{kj}^{(n)} b_{il}^{(0)} + \delta_{il}\, b_{ka}^{(n)} b_{aj}^{(0)} - b_{kj}^{(0)} b_{il}^{(n)} - \delta_{kj}\, b_{ia}^{(0)} b_{al}^{(n)} \right) \tag{4.12}$$

which shows that, in each representation where $b^{(0)}$ is a constant, the $b^{(1)}$ generators form a Lie subalgebra and the $b^{(n)}$ generators (for any given n) fall into representations of this Lie subalgebra. In what follows, we will consider only this type of representation.

4.1.1. Commutative subalgebras of the $\mathcal{S}_R$-algebra. We give in this short section some commutative subalgebras which will be used in the following.

Property 4.1. *Let us introduce*

$$t(k) = \mathrm{tr}(b(k)) = \sum_{n=0}^{\infty} k^{-n} t^{(n)} \tag{4.13}$$

$t(k)$ defines a commutative subalgebra of $\mathcal{S}_R$: $[t(k_1), t(k_2)] = 0$.

Proof. We take the trace in the auxiliary space 1 of the relation (4.3):

$$[t(k_1), b(k_2)] = \frac{2\mathrm{i}k_1 g}{k_1^2 - k_2^2}\,[b(k_2), b(k_1)]. \tag{4.14}$$

Then, taking again the trace we get

$$[t(k_1), t(k_2)] = \frac{2\mathrm{i}k_1 g}{k_1^2 - k_2^2}\;\mathrm{tr}\left([b(k_2), b(k_1)]\right). \tag{4.15}$$

From $[t(k_1), t(k_2)] = -[t(k_2), t(k_1)]$, we deduce $\mathrm{tr}([b(k_2), b(k_1)]) = 0$, which implies $[t(k_1), t(k_2)] = 0$. □

Property 4.2. *Let $\mathbb{M}$ be any constant matrix such that $\mathbb{M}^2 = \mathbb{I}$ and correspondingly*

$$\tilde{t}_M(k) = \mathrm{tr}\left(\mathbb{M}b(k)\right) = \sum_{n=0}^{\infty} k^{-n} \tilde{t}_M^{(n)} \tag{4.16}$$

$\tilde{t}_M(k)$ defines a commutative subalgebra of $\mathcal{S}_R$ which commutes with the one given in proposition 4.1:

$$[\tilde{t}_M(k_1), \tilde{t}_M(k_2)] = 0 \qquad \textit{and} \qquad [t(k_1), \tilde{t}_M(k_2)] = 0. \tag{4.17}$$

Proof. We start again with (4.3), multiply it by $\mathbb{M}_1 \equiv \mathbb{M} \otimes \mathbb{I}$, and then take the trace in the auxiliary space 1:

$$\begin{aligned}[]
[\tilde{t}_M(k_1), b(k_2)] &= \frac{\mathrm{i}g}{k_1 - k_2}\left(b(k_2)\mathbb{M}b(k_1) - b(k_1)\mathbb{M}b(k_2)\right) \\
&\quad + \frac{\mathrm{i}g}{k_1 + k_2}\,[b(k_2)b(k_1), \mathbb{M}] - \frac{g^2\;\mathrm{tr}(\mathbb{M})}{k_1^2 - k_2^2}\,[b(k_2), b(k_1)].
\end{aligned} \tag{4.18}$$

Then, after a product on the left by $\mathbb{M}$ and the trace, we get

$$\begin{aligned}[]
[\tilde{t}_M(k_1), \tilde{t}_M(k_2)] &= \frac{\mathrm{i}g}{k_1 - k_2}\;\mathrm{tr}\,\mathbb{M}\big(b(k_2)\mathbb{M}b(k_1) - \mathbb{M}b(k_1)\mathbb{M}b(k_2)\big) \\
&\quad - \frac{g^2\;\mathrm{tr}(\mathbb{M})}{k_1^2 - k_2^2}\;\mathrm{tr}\left(\mathbb{M}[b(k_2), b(k_1)]\right).
\end{aligned}$$

The left-hand side of the above expression is skew-symmetric in (k_1, k_2), while the right-hand side is symmetric, so that both sides are identically zero. This proves the first part of the assertion.

Now, taking the trace of (4.18), one gets

$$[\tilde{t}_M(k_1), t(k_2)] = \frac{\mathrm{i}g}{k_1 - k_2}\;\mathrm{tr}\left(b(k_2)\mathbb{M}b(k_1) - b(k_1)\mathbb{M}b(k_2)\right) \tag{4.19}$$

which proves that $[\tilde{t}_M(k_1), t(k_2)]$ is symmetric under the exchange $u \leftrightarrow v$. On the other hand, if one multiplies (4.14) by $\mathbb{M}$ and then takes the trace, one obtains

$$[t(k_1), \tilde{t}_M(k_2)] = \frac{2\mathrm{i}k_1 g}{k_1^2 - k_2^2}\;\mathrm{tr}\left(\mathbb{M}[b(k_2), b(k_1)]\right). \tag{4.20}$$

The antisymmetric part in (k_1, k_2) of the above expression shows that we have $\mathrm{tr}(\mathbb{M}[b(k_2), b(k_1)]) = 0$, which then proves the second assertion. □

4.2. Analysis of the consistency relations

We now turn to the consequences of the constraints (4.7) and (4.8). As far as $b^{(0)}$ is concerned, we have already seen that it must be central, but the consistency relation (4.7) also implies

$$\left(b^{(0)}\right)^2 = \mathbb{I}. \tag{4.21}$$

In the Fock representations $\mathcal{F}_B$ we consider, $b^{(0)}$ is a constant matrix and (4.21) shows that it can be diagonalized (by a constant $gl(N)$ matrix). Thus, up to a conjugation, one can suppose that we have

$$b^{(0)} = \mathbb{E} = \sum_{i=1}^{N} \epsilon_i \, E_{ii} \qquad \text{with} \quad \epsilon_i = \pm 1. \tag{4.22}$$

For convenience, we will use the notation

$$\begin{array}{ll} A = \{\alpha \text{ such that } \epsilon_\alpha = +1\} \subset [1, N] & \dim A = M \\ \bar{A} = \{\bar{\alpha} \text{ such that } \epsilon_{\bar{\alpha}} = -1\} \subset [1, N] & \dim \bar{A} = N - M. \end{array} \tag{4.23}$$

In the basis (4.22), the commutation relations with the Lie subalgebra generators are

$$[b_{ij}^{(1)}, b_{kl}^{(n)}] = \mathrm{i}g(\epsilon_i + \epsilon_j)\Big(\delta_{il}\, b_{kj}^{(n)} - \delta_{kj}\, b_{il}^{(n)}\Big) \tag{4.24}$$

which indicates that the Lie subalgebra depends on the choice of $\mathbb{E}$. Indeed, the consistency relation (4.7) together with the choice of $b^{(0)} = \mathbb{E}$ lead to

$$b_{\alpha\bar{\beta}}^{(1)} = 0 = b_{\bar{\alpha}\beta}^{(1)} \tag{4.25}$$

so that the Lie subalgebra in $\mathcal{S}_R$ is a $gl(M) \oplus gl(N-M)$ one.

More generally, the analysis of the consistency relations shows that $b_{\alpha\bar{\beta}}^{(2n+1)}$, $b_{\bar{\beta}\alpha}^{(2n+1)}$, $b_{\alpha\beta}^{(2n+1)}$ and $b_{\bar{\alpha}\bar{\beta}}^{(2n+1)}$ can be expressed in terms of the other generators, so that $\mathcal{S}_R$ is generated by

$$\mathcal{S}_R^{(2n)} = \left\{ b_{\alpha\bar{\beta}}^{(2n)},\ b_{\bar{\beta}\alpha}^{(2n)} \right\}_{(\alpha\in A,\, \bar{\beta}\in\bar{A})} = (M, \overline{N-M}) + (\overline{M}, N-M) \tag{4.26}$$

$$\mathcal{S}_R^{(2n+1)} = \left\{ b_{\alpha\beta}^{(2n+1)},\ b_{\bar{\alpha}\bar{\beta}}^{(2n+1)} \right\}_{(\alpha,\beta\in A,\, \bar{\alpha},\bar{\beta}\in\bar{A})} = (M^2, 0) + (0, (N-M)^2) \tag{4.27}$$

where we have indicated the decomposition in $gl(M) \oplus gl(N-M)$ representations.

5. The NLS reflection matrices

Now, we come to the explicit construction of the reflection matrices as defined by (2.19), (2.20). As already expressed in section 2, to each allowed B matrix (defined up to a $gl(N)$ conjugation, see below) is associated a Fock space representation $\mathcal{F}_B$ of the reflection algebra $(\mathcal{B}_R, I)$. Indeed the value of the operator $b(k)$ on the Fock space vacuum Ω is directly given by the matrix $B(k)$: see (2.21).

5.1. Classification of the B reflection matrices

Condition (2.22), with R again defined by (4.1), leads to the equations

$$[B(k_1), B(k_2)] = 0 \tag{5.1}$$

$$B_1(k_1)B_1(k_2) - B_2(k_2)B_2(k_1) = \frac{k_1 + k_2}{k_1 - k_2}\,(B_2(k_2)B_1(k_1) - B_2(k_1)B_1(k_2)) \tag{5.2}$$

where

$$B_1(k) = \sum_{i,j=1}^{N} B_i{}^j(k)\, E_{ij} \otimes \mathbb{I} \qquad \text{and} \qquad B_2(k) = \sum_{i,j=1}^{N} B_i{}^j(k)\, \mathbb{I} \otimes E_{ij}. \tag{5.3}$$

From (5.1), one immediately deduces that the $B(k)$ matrices ($\forall\, k$) can simultaneously put into a triangular form using a constant $Gl(N)$ matrix:

$$B(k) = U\, T(k)\, U^{-1} \qquad \text{with} \quad [T(k_1), T(k_2)] = 0. \tag{5.4}$$

Moreover, since we have

$$R_{12}(k_1 - k_2)U_1U_2 = U_2U_1R_{12}(k_1 - k_2) \tag{5.5}$$

the transformation (5.4) defines an automorphism of the whole algebra $\mathcal{B}_R$, as well as of the condition (2.11). Hence, we can suppose without any restriction that the matrices $B(k)$ are triangular.

Imposing $k_2 = -k_1$ in (5.2) implies

$$B(k)B(-k) = B(-k)B(k) = \rho(k)\mathbb{I} \tag{5.6}$$

where ρ is an even function which must be real in order to satisfy (2.20).

Then, a detailed study (see the appendix) of (5.2) leads to the following classification of B reflection matrices:

$$\text{Case } \rho(k) \neq 0 \qquad B(k) = \beta(k)\frac{\mathbb{I} + \mathrm{i}ak\,\mathbb{E}}{1 + \mathrm{i}ak} \tag{5.7}$$

$$B(k) = \beta(k)\,\mathbb{E} \tag{5.8}$$

$$B(k) = \beta(k)\,(\mathbb{I} + ak\mathbb{J}) \tag{5.9}$$

$$\text{Case } \rho(k) = 0 \qquad B(k) = \beta(k)\,\mathbb{J} \tag{5.10}$$

with the conditions

$$\mathbb{E}^2 = \mathbb{I} \qquad \mathbb{J}^2 = 0 \qquad a \in \mathbb{R}. \tag{5.11}$$

Note that,s because of the $Gl(N)$ invariance, $\mathbb{E}$ can be taken diagonal:

$$\mathbb{E} = \sum_{i=1}^{N} \epsilon_i\, E_{ii} \qquad \text{with} \quad (E_{ii})_{kl} = \delta_{ik}\delta_{il} \tag{5.12}$$

and $\mathbb{J}$ can be fixed to its Jordanian form:

$$\mathbb{J} = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_r \end{pmatrix} \qquad \text{with} \quad J_i = (0) \qquad \text{or} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} 1 \leqslant i \leqslant r. \tag{5.13}$$

Requiring the matrix $B(k)$ to obey the condition (2.20) leads to the supplementary relation:

$$\beta(k)^* = \beta(-k). \tag{5.14}$$

Let us remark that the solution (5.9) obeys (2.22), but not to the development (4.4), and as such should be rejected in this context. The other solutions can all be expanded in series of k^{-1} (provided $\beta(k)$ can be).

5.2. Boundary conditions associated with $B(k)$

From the algebraic study of the $\mathcal{S}_R$ algebra and its Fock representations, one can determine the boundary conditions obeyed by the physical fields $\Psi(x, t)$. For such a purpose we associate with each reflection matrix $B(k)$ a differential operator (in the variable x) D_B that will act on $\Psi(x, t)$. From (3.17), one infers that this operator will be fixed by its value on the eigenstates

$$\lambda_k(x) = \mathrm{e}^{\mathrm{i}kx} + B(k)\mathrm{e}^{-\mathrm{i}kx} \tag{5.15}$$

through the equation

$$D_B\,\lambda_k(x)|_{x=0} = 0. \tag{5.16}$$

We restrict our analysis to the physical condition

$$B(k)\cdot B(-k) = \mathbb{I}. \tag{5.17}$$

In all cases, the boundary condition for the physical field $\Psi(x,t)$ takes the form:

$$D_B\,\Psi(x,t)|_{x=0} = 0. \tag{5.18}$$

We specify below the operator D_B according to the classification (5.7)–(5.10).

5.2.1. Case of $B(k) = \beta(k)\mathbb{I}$. From the condition $\beta(k)\beta(-k) = 1$, one deduces that $\beta(k)$ can be rewritten as

$$\beta(k) = \frac{A(k)+\mathrm{i}}{A(k)-\mathrm{i}} \qquad \text{with} \quad A(k) \text{ real odd function.} \tag{5.19}$$

Defining the differential operator D_A through

$$(D_A f)(y) = \int \mathrm{d}x\,\hat{A}(x)\,f(x+y) \qquad \forall\, f\ \mathbb{C}\text{-function} \tag{5.20}$$

where $\hat{A}$ is the inverse Fourier transform of A. By definition of $\hat{A}$, we have

$$D_A e_p = A(p)e_p \qquad \text{where} \quad e_p(y) = \exp(\mathrm{i}py). \tag{5.21}$$

This implies

$$(D_A+\mathrm{i})\lambda_k = (A(k)+\mathrm{i})e_k + \beta(k)(A(-k)+\mathrm{i})e_{-k} = (A(k)+\mathrm{i})(e_k - e_{-k}). \tag{5.22}$$

In other words, one gets $(D_A+\mathrm{i})\lambda_k(x)|_{x=0} = 0$, and $D_B = D_A + \mathrm{i}$ is the required operator.

Let us remark that in the particular case advocated in (3.10)

$$\beta(k) = \frac{k-\mathrm{i}\eta}{k+\mathrm{i}\eta} \qquad \text{i.e.} \quad A(k) = \frac{k}{\eta} \tag{5.23}$$

we get $\hat{A}(x) = \frac{-\mathrm{i}}{\eta}\frac{\partial}{\partial x}$ and we recover the boundary condition (3.2), using the fact that $\partial_x f(x+y) = \partial_y f(x+y)$.

5.2.2. Case of $B(k) = \beta(k)\mathbb{E}$. As in the previous case, we introduce the odd function A defined in (5.19). The calculation is similar to the previous one, except for the matrix dependence. We define the differential operator D_A through (5.20). Then, one can choose

$$D_B = \frac{\mathbb{I}+\mathbb{E}}{2}(D_A+\mathrm{i}) + \frac{\mathbb{I}-\mathbb{E}}{2}(D_A+\mathrm{i})\partial_x. \tag{5.24}$$

This operator will obey the condition (5.16), because of the properties:

$$D_A e_k = A(k)e_k \qquad \text{and} \qquad D_A\partial_x e_k = \mathrm{i}kA(k)e_k \tag{5.25}$$

$$(\mathbb{I}+\mathbb{E})\mathbb{E} = \mathbb{I}+\mathbb{E} \qquad \text{and} \qquad (\mathbb{I}-\mathbb{E})\mathbb{E} = -(\mathbb{I}-\mathbb{E}) \tag{5.26}$$

which lead to

$$D_B\lambda_k = \frac{\mathbb{I}+\mathbb{E}}{2}(A(k)+\mathrm{i})(e_k - e_{-k}) + \frac{\mathbb{I}-\mathbb{E}}{2}\mathrm{i}k(A(k)+\mathrm{i})(e_k - e_{-k}). \tag{5.27}$$

It is then obvious that $D_B\lambda_k|_{x=0} = 0$.

5.2.3. Case of $B(k) = \beta(k)\frac{\mathbb{I}+iak\,\mathbb{E}}{1+iak}$. This case comes as a mixing of the two previous ones. We first rewrite B as

$$B(k) = \beta(k)\frac{\mathbb{I}+\mathbb{E}}{2} + \tilde{\beta}(k)\frac{\mathbb{I}-\mathbb{E}}{2} \qquad \text{with} \quad \tilde{\beta}(k) = \beta(k)\frac{1-\mathrm{i}ak}{1+\mathrm{i}ak} \tag{5.28}$$

and define $A(k)$ by (5.19) and $\tilde{A}(k)$ by

$$\tilde{\beta}(k) = \frac{\tilde{A}(k)+\mathrm{i}}{\tilde{A}(k)-\mathrm{i}} \qquad \text{i.e.} \quad \tilde{A}(k) = \frac{A(k)+ak}{1-akA(k)}. \tag{5.29}$$

The differential operators D_A and $D_{\tilde{A}}$ will be constructed as above, and it is straightforward to check that

$$D_B = \frac{\mathbb{I}+\mathbb{E}}{2}(D_A+\mathrm{i}) + \frac{\mathbb{I}-\mathbb{E}}{2}(D_{\tilde{A}}+\mathrm{i}) \tag{5.30}$$

obeys the relation (5.16).

6. Spontaneous symmetry breaking

In the Fock space representation $\mathcal{F}_B$ of the reflection algebra $\{\mathcal{B}_R, I\}$, once the matrix $B(k)$ is fixed, one knows all the operators $b^{(n)}$ which have non-vanishing value on Ω. Since the $\mathcal{S}_R$ algebra constitutes the symmetry algebra of our problem, we are exactly faced with a mechanism of spontaneous symmetry breaking for our reflection algebra, itself part of our $\mathcal{B}_R$ algebra. We present a generating function for the broken generators, and show that these generators form a commutative subalgebra of $\mathcal{S}_R$.

6.1. Case of $B(k) = \beta(k)\mathbb{I}$

We start with the reflection matrix:

$$B(k) = \beta(k)\mathbb{I} \qquad \text{with} \quad \beta(k) = \sum_{n\in J}\beta_n k^{-n} \qquad \beta_n \neq 0 \quad \forall n \in J \subset \mathbb{N}. \tag{6.31}$$

The vacuum expectation value of $b(k)$ is thus

$$\langle b_{ij}^{(p)}\rangle = \beta_p\delta_{ij} \;\Rightarrow\; \langle b_{ii}^{(p)} - b_{jj}^{(p)}\rangle = 0 \qquad \forall p. \tag{6.32}$$

Thus, the broken generators are all contained in[3]

$$\sum_{i=1}^{N} b_{ii}^{(p)} = \mathrm{tr}(b^{(p)}) \qquad \forall p \in J. \tag{6.33}$$

We first gather them into

$$t_B(u) = \sum_{p\in J} u^{-p}\,\mathrm{tr}(b^{(p)}). \tag{6.34}$$

Let us define the operator (depending on the new variable u and on the variable $x = k^{-1}$):

$$D_u^k = \sum_{p\in J}\frac{1}{p!}\left(u^{-1}\partial\right)^p \equiv \mathrm{d}(u^{-1}\partial) \qquad \text{with} \quad \partial = \frac{\partial}{\partial(k^{-1})} \qquad \mathrm{d}(x) = \sum_{p\in J}\frac{x^p}{p!}. \tag{6.35}$$

Then, from an obvious calculation, we have:

[3] We will ignore in our general approach the possibility of two broken generators of different grade, i.e. $\mathrm{tr}\, b^{(n)}$ and $\mathrm{tr}\, b^{(m)}$, to combine and form a third element, i.e. $\beta_m\, \mathrm{tr}\, b^{(n)} - \beta_n\, \mathrm{tr}\, b^{(m)}$, for which the symmetry is restored. A way of determining this last element (which algebraically annihilates the vacuum) would be to construct, when possible, a compensating transformation in the momentum space which preserves the correlation functions.

Property 6.1. *The generating function $t_B(u)$ is given by*

$$t_B(u) = \operatorname{tr}(D_u^k b(k))|_{k=\infty} = D_u^k\, t(k)|_{k=\infty} \tag{6.36}$$

where $t(k)$ has been defined in (4.13). It satisfies

$$t_B(u)\Omega = B(u)\Omega. \tag{6.37}$$

6.2. Case of $B(k) = \beta(k)\mathbb{E}$

Similarly, for

$$B(k) = \beta(k)\mathbb{E} \qquad \text{with} \quad \begin{cases} \beta(k) = \sum_{n\in J} \beta_n k^{-n} & \beta_n \neq 0 \quad \forall n \in J \subset \mathbb{N} \\ \mathbb{E} = \sum_{\alpha=1}^{M} E_{\alpha\alpha} - \sum_{\bar\alpha=M+1}^{N} E_{\bar\alpha\bar\alpha} = \sum_{i=1}^{N} \epsilon_i E_{ii} \end{cases} \tag{6.38}$$

we get the conditions:

$$\langle b_{ij}^{(p)} \rangle = \beta_p \epsilon_i \delta_{ij} \;\Rightarrow \langle \epsilon_i b_{ii}^{(p)} - \epsilon_j b_{jj}^{(p)} \rangle = 0 \qquad \forall p. \tag{6.39}$$

Then, the broken generators are of the form:

$$\sum_{i=1}^{N} \epsilon_i b_{ii}^{(p)} = \operatorname{tr}(\mathbb{E} b^{(p)}) \qquad \forall p \in J. \tag{6.40}$$

With the same calculation, as in the previous section, we obtain:

Property 6.2. *The generating function for the broken generators is given by*

$$t_B(u) = \sum_{p\in J} u^{-p} \operatorname{tr}(\mathbb{E} b^{(p)}) = \operatorname{tr}(\mathbb{E} D_u^k b(k))|_{k=\infty} = D_u^k\, \tilde t_E(k)|_{k=\infty} \qquad \forall u \tag{6.41}$$

with the same definition (6.35) of $D_u^k = \mathrm{d}(u^{-1}\partial)$. $\tilde t_E(k)$ has been defined in (4.16), here with $\mathbb{M} = \mathbb{E}$. $t_B(u)$ satisfies

$$t_B(u)\Omega = B(u)\Omega. \tag{6.42}$$

6.3. General case

Up to a redefinition of β and a, one can rewrite $B(k)$ defined in (5.7) as

$$B(k) = \frac{\beta(k)}{1+a}(\mathbb{E} + ak^{-1}\mathbb{I}) \tag{6.43}$$

so that the two previous cases are given by the limits $a \to 0$ or ∞. We have

$$\langle b_{ij}(k) \rangle = \frac{\beta(k)}{1+a}(\epsilon_i + ak^{-1})\delta_{ij} \tag{6.44}$$

which shows that

$$\langle b_{\alpha\alpha}(k) - b_{\beta\beta}(k) \rangle = 0 \qquad \forall\, \alpha, \beta = 1, \ldots, M \tag{6.45}$$

$$\langle b_{\bar\alpha\bar\alpha}(k) - b_{\bar\beta\bar\beta}(k) \rangle = 0 \qquad \forall\, \bar\alpha, \bar\beta = M+1, \ldots, N. \tag{6.46}$$

Once again, the broken generators are gathered in a generating function $t_B(u)$, and we get:

Property 6.3. *The generating function for broken generators is given by*

$$t_B(u) = \mathrm{tr}\left(D_u^k b(k)\right)\big|_{k=\infty} = \frac{1}{1+a}\left(\mathrm{d}(u^{-1}\partial)t(k) + a\hat{d}(u^{-1}\partial)\tilde{t}_E(k)\right)\big|_{k=\infty} \tag{6.47}$$

where now the matrix differential operator is

$$D_u^k = \frac{1}{1+a}\left(\mathbb{E}\,\mathrm{d}(u^{-1}\partial) + \mathbb{I}\,a\,\hat{\mathrm{d}}(u^{-1}\partial)\right) \qquad \textit{with} \quad \beta(k) = \sum_{p\in J}\beta_p k^{-p}.$$

As above

$$\mathrm{d}(x) = \sum_{p\in J}\frac{1}{p!}x^p \qquad \textit{and} \qquad \hat{\mathrm{d}}(x) = \sum_{p\in J}\frac{1}{(p+1)!}x^{p+1} \tag{6.48}$$

$t_B(u)$ *satisfies*

$$t_B(u)\Omega = B(u)\Omega. \tag{6.49}$$

Corollary 6.4. *The 'broken generators' form a commutative algebra:*

$$[t_B(u), t_B(v)] = 0. \tag{6.50}$$

Proof. Direct consequence of properties (4.1) and (4.2). □

Remark. In all cases, $t_B(u)$ is a scalar matrix, which means that we have at most only one broken generator at each level.

6.4. Example: $B(k) = \mathbb{I}$

We are in the particular case $\eta = 0$ of the conditions given in (3.2) and (3.3), so that the boundary conditions are here

$$\lim_{x\to\infty}\Phi(x,t) = 0 \qquad \text{and} \qquad \lim_{x\downarrow 0}\partial_x\Phi(x,t) = 0. \tag{6.51}$$

Due to the form of $B(k)$, it is obvious that none of the generators is broken. Hence, we are in the very specific situation where no spontaneous symmetry breaking occurs.

However, we remark that the boundary has an effect on the symmetry algebra: indeed, the Yangian symmetry which appears in the NLS hierarchy on the full line [19] is reduced here down to the 'smaller' $\mathcal{S}_R$ algebra.

7. The particular case $N = 2$

7.1. Characteristic of $N = 2$

The distinctive feature of $N = 2$ lies in the fact that the Pauli matrices, together with the identity matrix, form a basis of 2×2 matrices and that all these matrices obey the property 4.2. Indeed, one has:

Property 7.1. *For $N = 2$, $t(k)$ is central in the $\mathcal{S}_R$ algebra, and a generating system of this algebra is given by*

$$t(k) = \mathrm{tr}\; b(k) = b_{11}(k) + b_{22}(k) \qquad \textit{and} \qquad \tilde{t}_a(k) = \mathrm{tr}(\sigma_a b(k)) \qquad a = 1, 2, 3 \tag{7.52}$$

or, more explicitly,

$$\tilde{t}_1(k) = b_{12}(k) + b_{21}(k) \qquad \tilde{t}_2(k) = \mathrm{i}(b_{12}(k) - b_{21}(k)) \qquad \tilde{t}_3(k) = b_{11}(k) - b_{22}(k).$$

Proof. $\tilde{t}_a(k)$ are all generators obeying the property 4.2, which proves that $t(k)$ commutes with these generators. It is then enough to show that these generators form a generating system (together with $t(k)$) to prove that $t(k)$ is central (using also property 4.1). Using

$$\sigma_a \sigma_b = \delta_{ab}\,\mathbb{I} + \mathrm{i}\epsilon_{ab}{}^{c}\,\sigma_c \tag{7.53}$$

and $\operatorname{tr}\sigma_a = 0$, one gets

$$b(k) = \tfrac{1}{2}\left(t(k) + \sum_{a=1}^{3} \tilde{t}_a(k)\,\sigma_a\right) \tag{7.54}$$

showing therefore that $\{t(k), \tilde{t}_a(k),\ a = 1, 2, 3\}$ is a generating system (as $b_{ij}(k)$). □

The above property allows us to give a simpler commutation relation for the $\mathcal{S}_R$-algebra:

Property 7.2. *The commutation relations of the $\mathcal{S}_R$ algebra, for $N = 2$, are given by*

$$[t(k_1), t(k_2)] = 0 \qquad [t(k_1), \tilde{t}_a(k_2)] = 0 \tag{7.55}$$

$$[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = \frac{-2g\epsilon_{ab}{}^{c}}{k_1 - k_2}\,\frac{k_1\,t(k_1)\tilde{t}_c(k_2) - k_2\,t(k_2)\tilde{t}_c(k_1)}{k_1 + k_2 + \mathrm{i}g} \tag{7.56}$$

$$\tilde{t}_a(k_1)\tilde{t}_b(k_2) = \tilde{t}_a(k_2)\tilde{t}_b(k_1). \tag{7.57}$$

Proof. Equations (7.55) are just the rephrasing of the previous property. It remains to prove (7.56). We start with (4.18), for $\mathbb{M} = \sigma_a$, multiply it by σ_b and take the trace:

$$\begin{aligned}[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = {} & \frac{\mathrm{i}g}{k_1 - k_2}\,\operatorname{tr}(\sigma_b b(k_2)\sigma_a b(k_1) - \sigma_b b(k_1)\sigma_a b(k_2)) \\ & + \frac{\mathrm{i}g}{k_1 + k_2}\,\operatorname{tr}\big(\sigma_b[b(k_2)b(k_1),\ \sigma_a]\big).\end{aligned} \tag{7.58}$$

Using the expression (7.54), this can be rewritten as

$$\begin{aligned}4[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = {} & \frac{\mathrm{i}g}{k_1 - k_2}\,\operatorname{tr}(\sigma_b\sigma_d\sigma_a\sigma_c)\big(\tilde{t}_d(k_2)\tilde{t}_c(k_1) - \tilde{t}_d(k_1)\tilde{t}_c(k_2)\big) \\ & + \frac{\mathrm{i}g}{k_1 + k_2}\operatorname{tr}\big(\sigma_b[\sigma_c\sigma_d, \sigma_a]\big)\,\tilde{t}_c(k_2)\tilde{t}_d(k_1) \\ & + 4\mathrm{i}\epsilon_{ab}{}^{c}\left(\frac{\mathrm{i}g}{k_1 + k_2}\big(t(k_1)\tilde{t}_c(k_2) + t(k_2)\tilde{t}_c(k_1)\big)\right. \\ & \left. + \frac{\mathrm{i}g}{k_1 - k_2}\big(t(k_1)\tilde{t}_c(k_2) - t(k_2)\tilde{t}_c(k_1)\big)\right)\end{aligned} \tag{7.59}$$

with summation over repeated indices. A direct calculation shows

$$\operatorname{tr}\big(\sigma_b[\sigma_c\sigma_d, \sigma_a]\big) = 4\big(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}\big) \tag{7.60}$$

$$\operatorname{tr}\big(\sigma_b\sigma_d\sigma_a\sigma_c\big) = 2\big(\delta_{bd}\delta_{ac} + \delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}\big) \tag{7.61}$$

so that we get

$$\begin{aligned}[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = {} & -\frac{1}{2}\frac{\mathrm{i}g}{k_1 - k_2}\big([\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_a(k_2), \tilde{t}_b(k_1)]\big) \\ & + \frac{\mathrm{i}g}{k_1 + k_2}\big(\tilde{t}_b(k_2)\tilde{t}_a(k_1) - \tilde{t}_a(k_2)\tilde{t}_b(k_1)\big) \\ & + \frac{\mathrm{i}\epsilon_{ab}{}^{c}}{k_1^2 - k_2^2}\big(2\mathrm{i}gk_1\,t(k_1)\tilde{t}_c(k_2) - 2\mathrm{i}gk_2\,t(k_2)\tilde{t}_c(k_1)\big).\end{aligned}$$

Exchanging $(a, k_1) \leftrightarrow (b, k_2)$, the above equality leads to

$$\begin{aligned}[\tilde{t}_b(k_2), \tilde{t}_a(k_1)] = &-\frac{1}{2}\frac{\mathrm{i}g}{k_1 - k_2}\big([\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_a(k_2), \tilde{t}_b(k_1)]\big) \\ &+\frac{\mathrm{i}g}{k_1 + k_2}\big(\tilde{t}_a(k_1)\tilde{t}_b(k_2) - \tilde{t}_b(k_1)\tilde{t}_a(k_2)\big) \\ &-\frac{\mathrm{i}\epsilon_{ab}{}^c}{k_1^2 - k_2^2}\big(2\mathrm{i}gk_1\, t(k_1)\tilde{t}_c(k_2) - 2\mathrm{i}gk_2\, t(k_2)\tilde{t}_c(k_1)\big)\end{aligned}$$

so that

$$\begin{aligned}2[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] &= [\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_b(k_2), \tilde{t}_a(k_1)] \\ &= \frac{\mathrm{i}g}{k_1 + k_2}\big([\tilde{t}_b(k_2), \tilde{t}_a(k_1)] + [\tilde{t}_b(k_1), \tilde{t}_a(k_2)]\big) \\ &\quad -\frac{\mathrm{i}\epsilon_{ab}{}^c}{k_1^2 - k_2^2}\big(4\mathrm{i}gk_1\, t(k_1)\tilde{t}_c(k_2) - 4\mathrm{i}gk_2\, t(k_2)\tilde{t}_c(k_1)\big).\end{aligned}$$

Computing $[\tilde{t}_b(k_2), \tilde{t}_a(k_1)] + [\tilde{t}_b(k_1), \tilde{t}_a(k_2)]$ leads to

$$\begin{aligned}&\frac{k_1 - k_2 + \mathrm{i}g}{k_1 - k_2}\big([\tilde{t}_a(k_1), \tilde{t}_b(k_2)] + [\tilde{t}_a(k_2), \tilde{t}_b(k_1)]\big) \\ &\quad = \frac{4\mathrm{i}\epsilon_{ab}{}^c}{k_1^2 - k_2^2}\big(\mathrm{i}gk_1\, t(k_1)\tilde{t}_c(k_2) - \mathrm{i}gk_2\, t(k_2)\tilde{t}_c(k_1)\big)\end{aligned}$$

which proves (7.56).

On the other hand, starting from (7.59), one computes

$$[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_a(k_2), \tilde{t}_b(k_1)] = \frac{\mathrm{i}g}{k_1 + k_2}\big(\{\tilde{t}_a(k_1), \tilde{t}_b(k_2)\} - \{\tilde{t}_a(k_2), \tilde{t}_b(k_1)\}\big).$$

Then, using (7.56), one gets

$$[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = [\tilde{t}_a(k_2), \tilde{t}_b(k_1)] \tag{7.62}$$

$$\{\tilde{t}_a(k_1), \tilde{t}_b(k_2)\} = \{\tilde{t}_a(k_2), \tilde{t}_b(k_1)\} \tag{7.63}$$

which is equivalent to the second relation. □

Corollary 7.3. *The commutation relations with the Lie subalgebra generators are*

$$[\tilde{t}_a^{(1)}, \tilde{t}_b(k_2)] = -2g\epsilon_{ab}{}^c\, t^{(0)}\tilde{t}_c(k_2) \tag{7.64}$$

$$[t^{(1)}, \tilde{t}_b(k_2)] = 0. \tag{7.65}$$

Proof. One picks up the term k_1^{-1} in the previous commutators. □

7.2. Vacuum preserving algebra when $B(k) = \beta(k)\mathbb{I}$

In this case, the broken generators are all gathered in $t(k)$, and we can replace $t(k)$ (which is central) by its value $2\beta(k)$.

Thus, the vacuum preserving algebra is just the one generated by $\tilde{t}_a(k)$, $a = 1, 2, 3$. Its commutation relations are

$$[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = \frac{-4g\epsilon_{ab}{}^c}{(k_1 + k_2 + \mathrm{i}g)(k_1 - k_2)}\big(k_1\, \beta(k_1)\tilde{t}_c(k_2) - k_2\, \beta(k_2)\tilde{t}_c(k_1)\big). \tag{7.66}$$

Specifying to $m = n = 1$, we get

$$\big[\tilde{t}_a^{(1)}, \tilde{t}_b^{(1)}\big] = -4g\epsilon_{ab}{}^c\, \tilde{t}_c^{(1)} \tag{7.67}$$

which is an sl_2 Lie subalgebra.

Then, at the Lie subalgebra level, the spontaneous symmetry breaking is a $gl_2 \to sl_2$ one.

7.3. *Vacuum preserving algebra when* $B(k) = \beta(k)\sigma_3$

The broken generator is now $\tilde{t}_3(k)$ and its value is $2\beta(k)$.

Thus, the vacuum preserving algebra is now the one generated by $t(k)$, $\tilde{t}_1(k)$ and $\tilde{t}_2(k)$. It is indeed an algebra, as can be seen from the commutators:

$$
\begin{aligned}
&[t(k_1), t(k_2)] = 0 \qquad [t(k_1), \tilde{t}_a(k_2)] = 0 \qquad a = 1, 2 \\
&[\tilde{t}_1(k_1), \tilde{t}_1(k_2)] = 0 \qquad [\tilde{t}_2(k_1), \tilde{t}_2(k_2)] = 0 \\
&[\tilde{t}_1(k_1), \tilde{t}_2(k_2)] = \frac{-4g}{(k_1 + k_2 + \mathrm{i}g)(k_1 - k_2)} \big(k_1\, \beta(k_2) t(k_1) - k_2\, \beta(k_1) t(k_2) \big).
\end{aligned}
$$

However, since we are in the case $\mathbb{E} = \sigma_3 \neq \mathbb{I}$, the condition $b(k)b(-k) = \mathbb{I}$ implies in particular $b_{12}^{(1)} = b_{21}^{(1)} = 0$, or in our generating system $\tilde{t}_1^{(1)} = \tilde{t}_2^{(1)} = 0$. Thus, for the values $m = n = 1$, we get only one generator $t^{(1)}$, and one recognizes a gl_1 Lie subalgebra.

Then, at the Lie subalgebra level, the spontaneous symmetry breaking is a $gl_1 \oplus gl_1 \to gl_1$ one.

8. Outlook and conclusions

We proposed in this paper an algebraic approach for studying the symmetry content and the phenomenon of spontaneous symmetry breaking in integrable systems on the half line. The main tool is the quantum inverse scattering transform, where the familiar ZF algebra is replaced by the boundary algebra $\mathcal{B}_R$. The latter reflects the breakdown of translation invariance on the half line, which is codified by a set of boundary generators. These generators close a Sklyanin type subalgebra $\mathcal{S}_R \subset \mathcal{B}_R$ and commute with the Hamiltonians of the whole integrable hierarchy under consideration. For this reason $\mathcal{S}_R$ represents the central point of our investigation.

The basic, and actually unique, input of the scheme is the R-matrix, which describes the two-body scattering and determines, via the boundary Yang–Baxter equation, all reflection matrices respecting integrability. Each reflection matrix in turn fixes a boundary condition and therefore the dynamics and the symmetry of the system. The concrete example we have focused on is the $gl(N)$-NLS model, whose R-matrix is given by (4.1). If one considers this model on the whole line, the underlying symmetry algebra is the Yangian $Y(sl(N))$ [20]. The corresponding generators can be constructed [19] in terms of the associated ZF algebra. The latter admits a unique (up to unitary equivalence) Fock representation, whose vacuum is annihilated by all Yangian generators, i.e. the whole Yangian $Y(sl(N))$ is an exact symmetry of the theory. On the half line the situation is quite different. The counterpart of $Y(sl(N))$ is the Sklyanin algebra $\mathcal{S}_R$. The integrability preserving boundary conditions are encoded in the reflection matrices, which at the same time parametrize the inequivalent Fock representations of the underlying boundary algebra $\mathcal{B}_R$. The existence of such representations is crucial. They have no analogue in the ZF algebra and allow us to describe the physically inequivalent phases of the system on the half line. The generators of $\mathcal{S}_R$, which do not annihilate the vacuum of a given phase, are spontaneously broken in that phase. The relative Hamiltonian (2.24) has a gapless spectrum $[0, +\infty)$, as required by the nonrelativistic version [21] of Goldstone's theorem.

The framework developed in this paper can be applied in a more general context to the trigonometric and elliptic series of R-matrices as well. It will be interesting to investigate the structure of the associated $\mathcal{S}_R$ algebras, which will shed new light on the symmetry content of the corresponding integrable models on the half line.

Appendix. The reflection matrices

We have seen in section 5.1 that the reflection matrices $B(k)$ are simultaneously triangularizable. We present here the general solution to the equations:

$$[T(k_1), T(k_2)] = 0 \tag{A.1}$$

$$T_1(k_1)T_1(k_2) - T_2(k_2)T_2(k_1) = \frac{k_1 + k_2}{k_1 - k_2}\big(T_2(k_2)T_1(k_1) - T_2(k_1)T_1(k_2)\big) \tag{A.2}$$

where $T(k)$ is a triangular matrix, subject to the condition (5.6). Note that the equations (A.1) and (A.2) are invariant under multiplication of T by a scalar function $\beta(k)$, so that we can restrict the condition (5.6) to

$$T(k)T(-k) = T(-k)T(k) = \rho_0\, \mathbb{I} \qquad \text{with} \quad \rho_0 = \begin{cases} 0 & \text{(a)} \\ 1 & \text{(b).} \end{cases} \tag{A.3}$$

Two main cases occur: $T(k)$ diagonal or not.

A.1. Case of $T(k)$ diagonal

The condition (A.1) is then automatically satisfied, and one has to choose the condition (A.3(b)) to get $T(k)$ not null. Writing $T(k) = \mathrm{diag}(d_1(k), \ldots, d_N(k))$ and projecting equation (A.2) on a generic diagonal element leads to

$$(x - y)(q_{ij}(x)q_{ij}(y) - 1) = (x + y)(q_{ij}(x) - q_{ij}(y)) \tag{A.4}$$

with

$$q_{ij}(x) = \frac{d_i(x)}{d_j(x)} \qquad \forall\, i, j = 1, \ldots, N \tag{A.5}$$

where the condition $d_i(k) \neq 0$ is ensured by (A.3-b). The general solution to the equation

$$(x - y)(g(x)g(y) - 1) = (x + y)(g(x) - g(y)) \tag{A.6}$$

is given by

$$g(x) = \frac{1 + ax}{1 - ax}. \tag{A.7}$$

Using the particular form (A.5) of $q_{ij}(x)$, we get the solution

$$d_i(x) = d_1(x)r_i(x) \qquad \text{with} \quad r_i(x) = \frac{1 + cx}{1 - cx} \qquad \text{or} \qquad \pm 1\ \forall i \tag{A.8}$$

that is

$$T(k) = \beta(k)\,\frac{\mathbb{E} + cx\mathbb{E}'}{1 + cx} \tag{A.9}$$

where $\mathbb{E}$ and $\mathbb{E}'$ are diagonal matrices with ± 1 on the diagonal. Plugging this solution into equation (A.2), we get the constraint

$$c(\mathbb{E}_1 - \mathbb{E}_2)(\mathbb{E}'_1 + \mathbb{E}'_2) = 0 \tag{A.10}$$

the general solution of which leads to the solutions (5.7) and (5.8) (up to a scalar multiplication).

A.2. The case where $T(k)$ is not diagonal

One writes $T(k) = D(k) + S(k)$, where D is diagonal and S strictly triangular. Projecting equations (A.1) and (A.2) on the diagonal shows that $D(k)$ is of the form (5.7) and (5.8) or is zero, depending on the value of ρ_0.

A.2.1. The subcase $\rho_0 = 0$. Then, $D = 0$ and the equations for S are

$$(x - y)S(x)S(y) = 0 \tag{A.11}$$

$$(x + y)(S_1(x)S_2(y) - S_1(y)S_2(x)) = 0. \tag{A.12}$$

The solution is given by $S(x) = \beta(x)S_0$ with $S_0^2 = 0$. Using again the $gl(N)$ invariance, one can put the constant matrix S_0 into its Jordanian form, and one recovers (5.10).

A.2.2. The subcase $\rho_0 = 1$. Now, we have $D(x) = \mathbb{E} + cx^{-1}\mathbb{I}$, and one has to treat separately $c = 0$ and $c \neq 0$. The techniques are similar in both cases: as an illustrative example, we present here only $c = 0$ (i.e. $D(x) = \mathbb{E}$).

Plugging the form of $D(x)$ into (A.2), and taking the trace in the auxiliary space 2 we get

$$(x - y)(\mathbb{E}S(y) + S(x)\mathbb{E} + S(x)S(y)) = (x + y)\epsilon(S(x) - S(y)) \qquad \text{with} \quad \epsilon = \frac{1}{N}\,\mathrm{tr}(\mathbb{E}).$$

If $\epsilon^2 \neq 1$, one gets $S(x) = S_0$, constant matrix, and the equation for T shows that $T^2 = (\mathbb{E} + S_0)^2 = \mathbb{I}$, which implies that T is diagonalizable: we are back to section A.1.

If $\epsilon^2 = 1$, $\mathbb{E} = \epsilon\mathbb{I}$, and one gets as equations

$$2x(S_1(y) - S_2(y)) - 2y(S_1(x) - S_2(x)) = \epsilon(x + y)(S_1(x)S_2(y) - S_1(y)S_2(x))$$

$$[S(x), S(y)] = 0.$$

The solution is $S(x) = xS_0$ with $S_0^2 = 0$, due to (A.3), and we get the solution (5.9).

References

[1] Ghoshal S and Zamolodchikov A B 1994 *Int. J. Mod. Phys.* A **9** 3841
[2] Fring A and Köberle R 1994 *Nucl. Phys.* B **419** 647
Fring A and Köberle R 1994 *Nucl. Phys.* B **421** 159
[3] Fendley P and Saleur H 1994 *Nucl. Phys.* B **428** 681
[4] Saleur H, Skorik S and Warner N P 1995 *Nucl. Phys.* B **441** 421
[5] Bowcock P, Corrigan E, Dorey P E and Rietdijk R H 1995 *Nucl. Phys.* B **445** 469
[6] Corrigan E 1998 *Int. J. Mod. Phys.* A **13** 2709
[7] Liguori A and Mintchev M *Nucl. Phys.* 1998 B **522** 345
[8] Corrigan E 2000 Boundary bound states in integrable quantum field theory *Proc. Non-Perturbative Quantum Effects 2000*
(Corrigan E 2000 *Preprint* hep-th/0010094)
[9] Affleck I, Oshikawa M and Saleur H 2000 Boundary critical phenomena in $SU(3)$ spin chains *Preprint* cond-mat/0011454
[10] Liguori A, Mintchev M and Zhao L 1998 *Commun. Math. Phys.* **194** 569
[11] Cherednik V I 1984 *Theor. Math. Phys.* **61** 977
[12] Gattobigio M, Liguori A and Mintchev M 1998 *Phys. Lett.* B **428** 143
[13] Gattobigio M, Liguori A and Mintchev M 1999 *J. Math. Phys.* **40** 2949
[14] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[15] Zamolodchikov A B and Zamolodchikov A B 1979 *Ann. Phys., NY* **120** 253
[16] Faddeev L D 1980 *Sov. Sci. Rev.* C **1** 107
[17] Fokas A S 1989 *Physica* D **35** 167
[18] Freidel L and Maillet J M 1991 *Phys. Lett.* B **262** 278
[19] Mintchev M, Ragoucy E, Sorba P and Zaugg Ph 1999 *J. Phys. A: Math. Gen.* **32** 5885
[20] Murakami S and Wadati M 1996 *J. Phys. A: Math. Gen.* **29** 7903
[21] Swieca J A 1970 Glodstone's theorem and related topics *Cargèse Lectures in Phys.* vol 4, ed D Kasler (New York: Gordon and Breach)